Predictive Aiming for Moving Targets

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1 Introduction

Aiming for a moving target can be a non-trivial task. Here we consider three scenarios: a stationary target, a target with constant velocity, and a target with constant acceleration.

2 Aiming at Targets

For all scenarios, it is assumed that the projectile has a starting point P_S and a constant velocity P_V, with a magnitude called the *muzzle velocity*, denoted by M. Knowing that $\|P_V\| = M$, we may write P_V as $M \cdot \hat{P}_V$, where $\hat{P}_V$ is the direction vector (of unit length). The target has a starting point T_S. With all other values fixed, our goal is to calculate $\hat{P}_V$ and fire in that direction.

2.1 Stationary Target

Aiming for a stationary target is the most trivial case. Let $V = T_S - P_S$ be the vector between the projectile's starting point and the target, then $\hat{P}_V = \frac{V}{\|V\|}$. (The projectile will hit in $t = \frac{\|V\|}{M}$ seconds.)

2.2 Moving Target

Aiming for a target with a constant velocity is more complicated. We must first determine which times, if any, the target can be hit. With this information, we can calculate where the target will be at that time, and treat it as a stationary target.

First, we define two functions that determine the current position of the projectile and target:

$$P(t) = P_S + P_V t$$

$$T(t) = T_S + T_V t$$

where T_V is the target's velocity. Assume that the projectile will hit the target at time-of-impact t_I, and set these functions to be equal:

$$
\begin{aligned}
P(t_I) &= T(t_I) && (2.1)\\
P_S + P_V t_I &= T_S + T_V t_I && (2.2)\\
P_V t_I &= T_S - P_S + T_V t_I && (2.3)
\end{aligned}
$$

For ease of calculation, let $V = T_S - P_S$ like above:

$$
\begin{aligned}
P_V t_I &= V + T_V t_I && (2.4)\\
\|P_V t_I\| &= \|V + T_V t_I\| && (2.5)\\
\|P_V\| t_I &= \|V + T_V t_I\| && (2.6)
\end{aligned}
$$

Recall that $\|P_V\| = \left\|M \cdot \hat{P}_V\right\| = M$:

$$
\begin{aligned}
M t_I &= \|V + T_V t_I\| && (2.7)\\
(M t_I) &= \|V + T_V t_I\|^2 && (2.8)\\
(M t_I)^2 &= (V + T_V t_I) \cdot (V + T_V t_I) && (2.9)
\end{aligned}
$$

The dot product is distributive:

$$
\begin{aligned}
(M t_I)^2 &= V \cdot V + 2V \cdot (T_V t_I) + (T_V t_I) \cdot (T_V t_I) && (2.10)\\
(M t_I)^2 &= \|V\|^2 + 2V \cdot (T_V t_I) + \|T_V t_I\|^2 && (2.11)\\
M^2 t_I^2 &= \|V\|^2 + (2V \cdot T_V)\, t_I + \|T_V\|^2 t_I^2 && (2.12)\\
0 &= \left(\|T_V\|^2 - M^2\right) t_I^2 + (2V \cdot T_V)\, t_I + \|V\|^2 && (2.13)
\end{aligned}
$$

Let:

$$
\begin{aligned}
a &= M^2 - \|T_V\|^2\\
b &= -2V \cdot T_V\\
c &= -\|V\|^2
\end{aligned}
$$

then we have:

$$
a t_I^2 + b t_I + c = 0 \qquad (2.14)
$$

which can be solved using the quadratic formula. The discriminant is:

$$
\Delta = b^2 - 4ac
$$

If $\Delta < 0$, then the target cannot be hit (this can occur when the projectile travels slower than the target). If $\Delta = 0$, then the projectile can hit the target exactly once, at $t_I = \frac{-b}{2a}$. Otherwise, $\Delta > 0$ and the target can be hit at two different times (once early in the path, behind the target, and once later in its path). We want to shoot at the later time, so calculate t_0 and t_1 and take the maximum:

$$
\begin{aligned}
t_0 &= \frac{-b + \sqrt{\Delta}}{2a} && (2.15)\\
t_1 &= \frac{-b - \sqrt{\Delta}}{2a} && (2.16)\\
t_I &= \sup(t_0, t_1) && (2.17)
\end{aligned}
$$

From here, we can calculate the target's future position at time of impact as $T(t_I)$, and fire in that direction as before:

$$
\begin{aligned}
V &= T(t_I) - P_S && (2.18)\\
\hat{P}_V &= \frac{V}{\|V\|} && (2.19)
\end{aligned}
$$

2.3 Accelerating Target

Like before, we'll need to calculate when the target can be hit. Define two new functions that determine the current position of the projectile and target:

$$P(t) = P_S + P_V t$$

$$T(t) = T_S + T_V t + \left(\frac{T_A}{2}\right) t^2$$

where T_A is the target's acceleration. For readability, let $\bar{T}_A = \frac{T_A}{2}$. Assume that the projectile will hit the target at time-of-impact t_I, and set these functions to be equal:

$$
\begin{aligned}
P(t_I) &= T(t_I) && (2.20)\\
P_S + P_V t_I &= T_S + T_V t_I + \bar{T}_A t_I^2 && (2.21)\\
P_V t_I &= T_S - P_S + T_V t_I + \bar{T}_A t_I^2 && (2.22)
\end{aligned}
$$

Again let $V = T_S - P_S$:

$$\begin{aligned}
P_V t_I &= V + T_V t_I + \bar{T}_A t_I^2 && (2.23)\\
\|P_V t_I\| &= \left\|V + T_V t_I + \bar{T}_A t_I^2\right\| && (2.24)\\
M t_I &= \left\|V + T_V t_I + \bar{T}_A t_I^2\right\| && (2.25)\\
(M t_I)^2 &= \left\|V + T_V t_I + \bar{T}_A t_I^2\right\|^2 && (2.26)\\
(M t_I)^2 &= \|V\|^2 + \|T_V t_I\|^2 + \left\|\bar{T}_A t^2\right\|^2 + 2V \cdot (T_V t_I) + 2V \cdot \left(\bar{T}_A t_I^2\right) + 2\,(T_V t_I) \cdot \left(\bar{T}_A t_I^2\right) && (2.27)\\
M^2 t_I^2 &= \|V\|^2 + \|T_V\|^2 t_I^2 + \left\|\bar{T}_A\right\|^2 t_I^4 + (2V \cdot T_V)\, t_I + \left(2V \cdot \bar{T}_A\right) t_I^2 + \left(2T_V \cdot \bar{T}_A\right) t_I^3 && (2.28)\\
0 &= \left\|\bar{T}_A\right\|^2 t_I^4 + \left(2T_V \cdot \bar{T}_A\right) t_I^3 + \left(\|T_V\|^2 + 2V \cdot \bar{T}_A - M^2\right) t_I^2 + (2V \cdot T_V)\, t_I + \|V\|^2 && (2.29)
\end{aligned}$$

Let:

$$\begin{aligned}
a &= \left\|\bar{T}_A\right\|^2\\
b &= 2T_V \cdot \bar{T}_A\\
c &= \|T_V\|^2 + 2V \cdot \bar{T}_A - M^2\\
d &= 2V \cdot T_V\\
e &= \|V\|^2
\end{aligned}$$

then we have:

$$a t_I^4 + b t_I^3 + c t_I^2 + d t_I + e = 0 \qquad (2.30)$$

which can be solved using quartic formula solving methods.[1] Like before, find the root with the greatest time-to-impact, calculate the target's future position at time of impact as $T(t_I)$, and fire in that direction.

2.4 Non-constant Acceleration

It is possible to extend the above calculations to allow non-constant accelerations. However, this raises the degree of the resulting polynomial to $d \geq 5$, which prevents its solution from being expressed algebraically.[2] Solving such an equation requires root-finding approximations, and is outside the scope of this paper.

[1] These are exact, but too complicated to list here

[2] This is known as the Abel-Ruffini theorem.